Generating Invariant Manifolds with the Parameterization Method

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Introduction

This paper is meant to serve as an intuitive way to understand and apply the parameterization method to generate invariant flows and maps.

Conceptual Understanding of the Invariance Equation

The general invariance equation for maps is

$$F(P(\sigma)) = P(R(\sigma)), \tag{1}$$

where F is the map, $P(\sigma) = \begin{pmatrix} x(\sigma) \\ y(\sigma) \end{pmatrix}$ is the parameterization, σ is the parameter, and R is the internal dynamics (or rule) that maps one value of σ to another. If we consider one value of σ, this invariance equation states that "Applying the map to a point $P(\sigma)$ on the manifold returns another point $P(R(\sigma))$ on the manifold." This is the precise definition of invariance.

Near a fixed point, motion along a one-dimensional invariant manifold expands or contracts by a factor λ due to the map F. The parameter σ labels points on the manifold; to match the rate of change determined by the map, we update the parameter by the same factor:

$$R(\sigma) = \lambda\sigma.$$

This ensures that applying the map to the point labeled by σ gives the point labeled by $\lambda\sigma$:

$$F(P(\sigma)) = P(\lambda\sigma).$$

In this way, the parameterization changes at the same rate that the map stretches or compresses distances along the manifold.

Finally, we set $P(0) = p$, where p is the fixed point of the invariant manifold. This centers the parameterization around the fixed point and greatly reduces calculations. We also set $P'(0) = \xi$, so that the manifold is tangent to the eigenvector at the fixed point.

The invariance equation for flows works in much the same way, so we will not discuss it conceptually for the sake of brevity.

1D Manifolds (Maps in Two Dimensions)

Let us apply the invariance equation $F(P(\sigma)) = P(\lambda\sigma)$ to the Hénon Map, defined by

$$F(x, y) = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}.$$

We assume that $P(\sigma)$ will be of the form

$$P(\sigma) = \sum_{n=0}^{\infty} p_n \sigma^n = \sum_{n=0}^{\infty} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \sigma^n.$$

We seek a way to calculate the coefficients p_n to find an approximation for our manifold. We can solve for the fixed point of the Hénon Map by solving:

$$F(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then, we calculate the eigenvalues and eigenvectors by finding the Jacobian matrix of the map. As previously stated, $P(0)$ and $P'(0)$ are equal to the fixed point and eigenvector, respectively. These values determine p_0 and p_1:

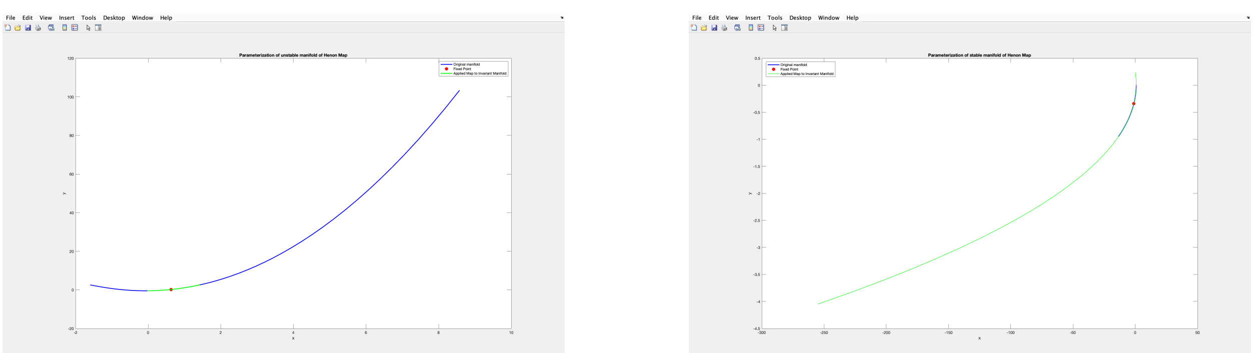
$$P(\sigma) = p_0 + p_1\sigma + p_2\sigma^2 + \cdots \implies P(0) = p_0,$$
$$P'(\sigma) = p_1 + 2p_2\sigma + 3p_3\sigma^2 \cdots \implies P'(0) = p_1.$$

By plugging our parameterization $P(\sigma)$ back into the invariance equation and rearranging terms, we arrive at the general equation for calculating the coefficients p_n:

$$[DF(p_0) - \lambda^n \mathrm{Id}]p_n = \begin{pmatrix} \sum_{k=1}^{n-1} a u_{n-k} u_k \\ 0 \end{pmatrix}.$$

By taking the inverse of the matrix $[DF(p_0) - \lambda^n \mathrm{Id}]$, we can calculate p_n in terms of $p_{n-1}, p_{n-2}, \ldots, p_0$.

To check that the manifold is truly invariant, one can apply the map to the manifold. Because the manifold is invariant, the mapped manifold should align with the original, though there will be a scale factor.



(a) Unstable Manifold

(b) Stable Manifold

Figure 1: Unstable and stable manifolds of the Hénon Map at $(x^*, y^*) = (0.63, 0.19)$.

1D Manifolds (Flows in Three Dimensions)

The invariance equation for one-dimensional invariant flows is

$$\sigma\lambda\frac{d}{d\sigma}P(\sigma) = F(P(\sigma)),$$

where $P(0) = p_0$ and $P'(0) = \xi$. Let us consider the Lorenz system:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases}$$

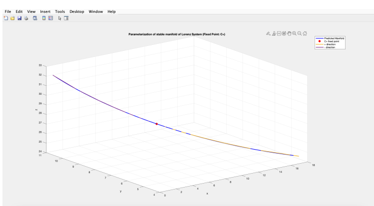
Again, we guess that $P(\sigma) = \sum_{n=0}^{\infty} p_n\sigma^n$. Substituting this into the invariance equation and solving for p_n gives:

$$[DF(p_0) - n\lambda\mathrm{Id}]p_n = R_n,$$

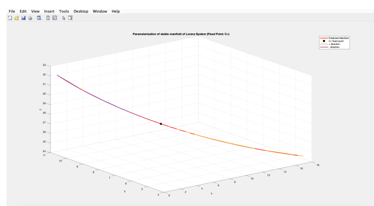
where

$$R_n = \begin{pmatrix} 0 \\ \sum_{k=1}^{n-1} a_{n-k}c_k \\ -\sum_{k=1}^{n-1} a_{n-k}b_k \end{pmatrix}.$$

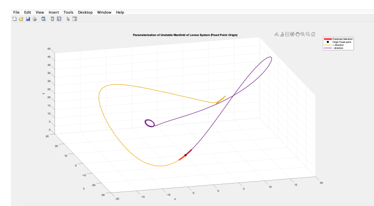
To check the validity of the manifolds, one can use backwards integration. Applying the parameterization method to the Lorenz System generates the following 1D manifolds:



(a) Stable at $C+$



(b) Stable at $C-$



(c) Unstable at Origin

Figure 2: 1D Manifolds of the Lorenz System.

2D Manifolds (Flows in Three Dimensions)

The invariance equation for a two-dimensional flow is given by:

$$(\lambda_1\sigma_1)\frac{\partial}{\partial\sigma_1}P(\sigma_1,\sigma_2) + (\lambda_2\sigma_2)\frac{\partial}{\partial\sigma_2}P(\sigma_1,\sigma_2) = F(P(\sigma_1,\sigma_2)), \quad (2)$$

where λ_1, λ_2 are eigenvalues at the fixed point p_0. We seek a bivariate power series:

$$P(\sigma_1,\sigma_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{n,m}\sigma_1^n\sigma_2^m,$$

where $P(0,0) = p_0$, $P(1,0) = \xi_1$, and $P(0,1) = \xi_2$. Collecting terms of order $\sigma_1^n\sigma_2^m$ leads to the homological equation:

$$[DF(p_0) - (n\lambda_1 + m\lambda_2)\mathrm{Id}]p_{n,m} = R_{n,m}. \quad (3)$$

The term $R_{n,m}$ is defined via the convolution:

$$(a\hat{*}b)_{n,m} = \left(\sum_{j=0}^{n} \sum_{k=0}^{m} a_{n-j,m-k} b_{j,k} \right) - a_{0,0} b_{n,m} - a_{n,m} b_{0,0}.$$

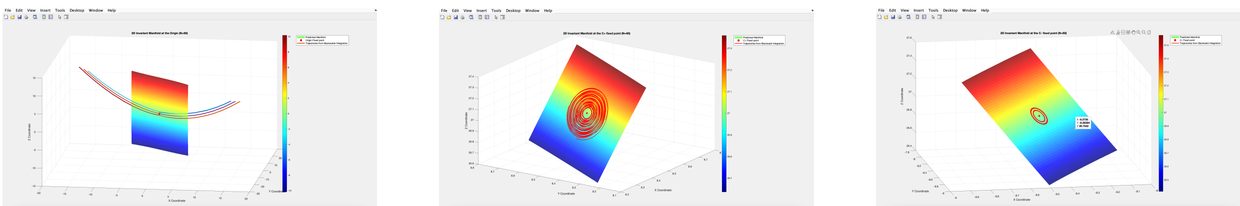
Notice that both 1D and 2D cases involve similar matrices multiplying $p_{n,m}$ and utilize convolutions in the R vector.

In many dynamical systems, including the Lorenz system, eigenvalues λ_1 and λ_2 may be a complex conjugate pair, $\lambda_{1,2} = \alpha \pm i\omega$. To ensure the final manifold is embedded in $\mathbb{R}^3$, we define:

$$\sigma_1 = u + iv, \quad \sigma_2 = u - iv.$$

By choosing $p_{1,0}$ and $p_{0,1}$ to be complex conjugates, all subsequent coefficients $p_{n,m}$ and $p_{m,n}$ will also be conjugates, allowing the imaginary parts to cancel out exactly.

To check the validity, one can use backwards integration or plot the decay of the coefficients at each order $k = n + m$. The objective is to scale the eigenvectors so that coefficients decrease to a sufficiently small ϵ.



(a) Stable at Origin (b) Unstable at $C+$ (c) Unstable at $C-$

Figure 3: 2D Manifolds from the Parameterization Method.